On PA Type Incomplete Block Designs
and PAB Type Rectangular Designs

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Abstract

Kurkjian and Zelen [1963] introduced a structural property, which was designated as Property A, of the (block) incidence matrix N associated with a class of block and direct product experiment designs. Extending this idea, Zelen and Federer [1964] introduced two structural properties, which were designated as Property A and Property B, associated with experiment designs for two-way elimination of heterogeneity, such as a k-row by b-column rectangular experiment design, and direct product designs. Property A is associated with the column (block) incidence matrix N and Property B is associated with the row incidence matrix $\tilde{N}$. These authors made use of the notation and the operations of the calculus for factorial arrangements introduced by Kurkjian and Zelen [1962]. The block designs possessing Property A will be designated as PA type designs in this paper. Likewise, those two-way elimination of heterogeneity designs possessing both Property A and Property B will be designated as PAB type designs.

The purpose of this paper is to present results associated with PA and PAB type designs. After a presentation of notation and definitions in the second section, results by the above authors on PA and PAB type designs are discussed. In the third section, a measure of efficiency for PA and for PAB type designs is presented. This measure makes use of the nonzero characteristic roots of the reduced normal equations matrix; a solution for these roots is given for PA and PAB type designs. In the last section, some relationships between partially balanced incomplete block (PBIB) designs and PA type designs are discussed; a detailed discussion of two-associate class PBIB designs is given in terms of their relation to PA type designs.

This work was partially supported by PHS Research Grant 5-R01-GM-05900.

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1. Summary and Introduction

Kurkjian and Zelen [1963] introduced a structural property, which was designated as Property A, of the (block) incidence matrix N associated with a class of block and direct product experiment designs. Extending this idea, Zelen and Federer [1964] introduced two structural properties, which were designated as Property A and Property B, associated with experiment designs for two-way elimination of heterogeneity, such as a k-row by b-column rectangular experiment design, and direct product designs. Property A is associated with the column (block) incidence matrix N and Property B is associated with the row incidence matrix $\tilde{N}$. These authors made use of the notation and the operations of the calculus for factorial arrangements introduced by Kurkjian and Zelen [1962]. The block designs possessing Property A will be designated as PA type designs in this paper. Likewise, those two-way elimination of heterogeneity designs possessing both Property A and Property B will be designated as PAB type designs.

The purpose of this paper is to present results associated with PA and PAB type designs. After a presentation of notation and definitions in the second section, results by the above authors on PA and PAB type designs are discussed. In the third section, a measure of efficiency for PA and for PAB type designs is presented. This measure makes use of the nonzero characteristic roots

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This work was partially supported by PHS Research Grant 5-R01-GM-05900.

of the reduced normal equations matrix; a solution for these roots is given for PA and PAB type designs. In the last section, some relationships between partially balanced incomplete block (PBIB) designs and PA type designs are discussed; a detailed discussion of two-associate class PBIB designs is given in terms of their relation to PA type designs.

2. Preliminaries

2.1. Notations and a basic theorem

The following notation will be used:

$\mathbf{1}_{m_i}$: $m_i \times 1$ column vector having all elements unity,

$J_{m_i} = \mathbf{1}_{m_i} \mathbf{1}'_{m_i}$: $m_i \times m_i$ matrix with all elements unity,

I_{m_i} : $m_i \times m_i$ identity matrix,

$$I_i^{x_i} = \begin{cases} \mathbf{1}_{m_i} & \text{if } x_i = 0 \\ I_{m_i} & \text{if } x_i = 1 \end{cases} .$$

$$D_i^{\delta_i} = \begin{cases} I_{m_i} & \text{if } \delta_i = 0 \\ J_{m_i} & \text{if } \delta_i = 1 \end{cases} .$$

$$M_i = m_i I_{m_i} - J_{m_i} .$$

$$M_i^{x_i} = \begin{cases} \mathbf{1}'_{m_i} & \text{if } x_i = 0 \\ M_i & \text{if } x_i = 1 \end{cases} .$$

The direct product or Kronecker product of $M_i^{x_i}$ and $M_j^{x_j}$ will be written as $M_i^{x_i} \otimes M_j^{x_j}$ and in general, the joint direct product of n $M_i^{x_i}$ $(i=1,2,\ldots,n)$ will be written as

$$\prod_{i=1}^{n} \otimes M_i^{x_i} .$$

From the above definition of matrices, the following results are easily verified:

$$M_i^{x_i} D_i^{\delta_i} = m_i^{(1-x_i)\delta_i}(1-x_i\delta_i)M_i^{x_i} \text{ for } x_i = 0,1 \text{ and } \delta_i = 0,1 \tag{2.1}$$

$$I_i^{x_i} M_i^{x_i} = m_i x_i I_{m_i} + (-1)^{x_i} J_{m_i} \quad \text{ for } x_i = 0,1 \tag{2.2}$$

$$D_i^{\delta_i} I_i^{x_i} M_i^{x_i} = m_i^{(1-x_i)\delta_i}(1-x_i\delta_i) I_i^{x_i} M_i^{x_i} \text{ for } \delta_i = 0,1 \text{ and } x_i = 0,1 \,. \tag{2.3}$$

Now, we shall prove the following lemma.

<u>Lemma 2.1</u>

$$\sum_{s=1}^{n} \sum_{x_1+x_2+\dots+x_n=s} \prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i} = v_n I_{v_n} - J_{v_n} \,, \tag{2.4}$$

<u>where</u> $v_n = \prod_{i=1}^{n} m_i$ <u>and</u> $x_i = 0$ <u>or</u> 1 <u>for</u> <u>all</u> $i = 1,2,\dots,n$.

<u>Proof</u>: To prove this lemma, note first that it is clearly true for $n=1$. Now assume that for some k,

$$\sum_{s=1}^{k} \sum_{x_1+x_2+\dots+x_k=s} \prod_{i=1}^{k} \otimes I_i^{x_i} M_i^{x_i} = v_k I_{v_k} - J_{v_k} \,.$$

Then,

$$\sum_{s=1}^{k+1} \sum_{x_1+x_2+\dots+x_n=s} \prod_{i=1}^{k+1} \otimes I_i^{x_i} M_i^{x_i}$$

$$= \sum_{s=1}^{k} \sum_{x_1+x_2+\dots+x_k=s} \prod_{i=1}^{k} \otimes I_i^{x_i} M_i^{x_i} \otimes J_{m_{k+1}}$$

$$+ \sum_{s=1}^{k+1} \sum_{x_1+x_2+\cdots+x_k=s-1} \prod_{i=1}^{k} \otimes I_i^{x_i} M_i^{x_i} \otimes (m_{k+1} I_{m_{k+1}} - J_{m_{k+1}})$$

$$= (v_k I_{v_k} - J_{v_k}) \otimes J_{m_{k+1}} + J_{v_k} \otimes (m_{k+1} I_{m_{k+1}} - J_{m_{k+1}})$$

$$+ (v_k I_{v_k} - J_{v_k}) \otimes (m_{k+1} I_{m_{k+1}} - J_{m_{k+1}})$$

$$= v_{k+1} I_{v_{k+1}} - J_{v_{k+1}} .$$

Thus, the formula is true for k+1. By the induction principle, this proves the lemma for all natural numbers.

The above lemma may be written more concisely by letting $x=(x_1,x_2,\ldots,x_n)$ be an n-digit binary number and by writing

$$I^x M^x = \prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i} .$$

Then, lemma 2.1 may be written as

$$\sum{}' I^x M^x = vI_v - J_v ,$$

where the summation $\sum'$ is over all nonzero n digit binary numbers x and $v = \prod_{i=1}^{n} m_i$.

From the results in the paper by Kurkjian and Zelen [1963], the following theorem can be formulated:

Theorem 2.1

Let

$$C = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\delta_2,\dots,\delta_n)\left(\prod_{i=1}^{n} \otimes D_i^{\delta_i}\right) \right\}. \tag{2.5}$$

where $\delta_i=0$ or 1 for $i=1,2,\dots,n$, and $g(\delta_1,\delta_2,\dots,\delta_n)$ are constants, and suppose

$CJ_v = 0$, $v = \prod_{i=1}^{n} m_i$. Then,

$$C^{+} = \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\dots+x_n=s} \frac{\prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i}}{rvE(x_1,x_2,\dots,x_n)} \right\}, \tag{2.6}$$

where $rE(x_1,x_2,\dots,x_n) = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\delta_2,\dots,\delta_n) \prod_{i=1}^{n} m_i^{(1-x_i)\delta_i}(1-x_i\delta_i) \right\}$,

is a generalized inverse of C if $E(x_1,x_2,\dots,x_n) \neq 0$ for all nonzero n-digit binary numbers $x = (x_1,x_2,\dots,x_n)$.

Proof:

$$M^{x}C = \left(\prod_{i=1}^{n} \otimes M_i^{x_i}\right)\left(\sum_{s=0}^{n}\left\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\delta_2,\dots,\delta_n)\left(\prod_{i=1}^{n} \otimes D_i^{\delta_i}\right)\right\}\right)$$

$$= \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\delta_2,\dots,\delta_n)\left(\prod_{i=1}^{n} \otimes M_i^{x_i} D_i^{\delta_i}\right) \right\}$$

$$= \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\delta_2,\dots,\delta_n) \prod_{i=1}^{n} \otimes m_i^{(1-x_i)\delta_i} M_i^{x_i} \right\}$$

$$= rE(x_1,x_2,\dots,x_n) \prod_{i=1}^{n} \otimes M_i^{x_i} = rE(x)M^{x} .$$

Then, since $C^+ = \sum' \frac{I^x M^x}{rvE(x)}$,

$$C^+C = \left[\sum' \frac{I^x M^x}{rvE(x)}\right]C = \sum' \frac{I^x M^x C}{rvE(x)} = \frac{1}{v}\sum' I^x M^x \ .$$

From lemma 2.1, C^+C may be written as:

$$C^+C = \frac{1}{v}(vI_v - J_v) = I_v - \frac{1}{v}J_v \ . \tag{2.7}$$

Hence,

$$CC^+C = C \quad \text{since } CJ_v = 0 \ . \tag{2.8}$$

Next,

$$C^+CC^+ = C^+ - \frac{1}{v}J_v C^+ \ .$$

Also,

$$J_v C^+ = \left(\prod_{i=1}^{n} \otimes J_{m_i}\right)\left(\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\ldots+x_n=s} \frac{\prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i}}{rvE(x_1,x_2,\ldots,x_n)}\right\}\right)$$

$$= \sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\ldots+x_n=s} \frac{\prod_{i=1}^{n} \otimes J_{m_i} I_i^{x_i} M_i^{x_i}}{rvE(x_1,x_2,\ldots,x_n)}\right\} = 0 \ ,$$

since $J_{m_i} I_i^{x_i} M_i^{x_i} = 0$ for $x_i = 1$ and $(x_1,x_2,\ldots,x_n) \neq (0,0,\ldots,0)$.

Therefore, $C^+CC^+ = C^+$. (2.9)

From (2.7), we note that C^+C is a symmetric matrix; then

$$(C^+C)' = C^+C \ . \tag{2.10}$$

Now, from (2.3) and lemma 2.1,

$$CC^+ = C\left[\sum' \frac{I^x M^x}{rvE(x)}\right]$$

$$= \frac{1}{v} \sum{}' I^x M^x = I_v - J_v/v \quad .$$

Then, $(CC^+)' = CC^+$. This completes the proof. (2.11)

2.2. PA type incomplete block designs.

Consider that the design consists of b blocks containing k treatments each and that each of the v treatments is replicated r times, and suppose $v = \prod_{i=1}^{n} m_i$, $n \geq 2$, where each m_i is a prime number. Let Y_{ij} be the yield of the i^{th} treatment in the j^{th} block and assume the usual fixed model

$$E(Y_{ij}) = \mu + t_i + b_j \ , \ i=1,2,\ldots,v \ ; \ j=1,2,\ldots,b \ . \tag{2.12}$$

where t_i is the (fixed) effect of the i^{th} treatment, b_j is the (fixed) effect of the j^{th} block and μ is the general mean effect. The t_i and b_j are assumed to satisfy the relations:

$$\sum_{i=1}^{v} t_i = 0 \ ; \ \sum_{j=1}^{b} b_j = 0 \ . \tag{2.13}$$

We further assume that $\{Y_{ij}\}$ are uncorrelated normal variates with common variance σ^2 .

Let $N = (n_{ij})$, $i=1,2,\ldots,v$; $j=1,2,\ldots,b$ be the incidence matrix of the design, where $n_{ij} = 0$ or 1 depending on whether the i^{th} treatment is absent or present in the j^{th} block. Then, it is well known that the reduced normal equations for estimating the treatment effect vector $t = (t_1,t_2,\ldots,t_v)'$ may be written as:

$$C\hat{t} = Q \ , \tag{2.14}$$

where $C = rI_v - NN'/k$, $Q = T - NB/k$;

$$T = (T_1,T_2,\ldots,T_v)' , \quad T_i = \sum_{j=1}^{b} n_{ij}Y_{ij} ;$$

$$B = (B_1,B_2,\ldots,B_b)' , \quad B_j = \sum_{i=1}^{v} n_{ij}Y_{ij} . \tag{2.15}$$

The solution of (2.14) is $\hat{t} = C^{+}Q$, where C^{+} is a generalized inverse of C .

Kurkjian and Zelen [1963] introduced a structural property of the design which was related to the block (or column) incidence matrix N of the design. This structural property was termed Property (A) and was defined as follows:

A block design will be said to have Property (A) or will be called a PA type design if

$$\text{(A)} \quad NN' = \sum_{s=0}^{n}\Big\{\sum_{\delta_1+\delta_2+\ldots+\delta_n=s} h(\delta_1,\delta_2,\ldots,\delta_n)\ \big(\prod_{i=1}^{n} \otimes\, D_i^{\delta_i}\big)\Big\} = \sum_{\delta} h(\delta)D^{\delta}, \tag{2.16}$$

where $\delta_i = 0$ or 1 for $i = 1,2,\ldots,n$, and $h(\delta_1,\delta_2,\ldots,\delta_n)$ are constants. In this case,

$$C = \sum_{s=0}^{n}\Big\{\sum_{\delta_1+\delta_2+\ldots+\delta_n=s} g(\delta_1,\delta_2,\ldots,\delta_n)\ \big(\prod_{i=1}^{n} \otimes\, D_i^{\delta_i}\big)\Big\} = \sum_{\delta} g(\delta)D^{\delta} , \tag{2.17}$$

where $g(0,0,\ldots,0) = r - \frac{1}{k}h(0,0,\ldots,0)$ and $g(\delta_1,\delta_2,\ldots,\delta_n) = -\frac{1}{k}h(\delta_1,\delta_2,\ldots,\delta_n)$ for nonzero vectors $\delta = (\delta_1,\delta_2,\ldots,\delta_n)$. Also, it is well known that $CJ_v = 0$. Therefore, from Theorem 2.1, we obtain the following solution for equation (2.14):

$$\hat{t} = \sum_{s=0}^{n}\Big\{\sum_{x_1+x_2+\ldots+x_n=s} \frac{\big(\prod_{i=1}^{n} \otimes\, I_i^{x_i}M_i^{x_i}\big)}{rvE(x_1,x_2,\ldots,x_n)}\Big\}\, Q = \Big[{\sum}' \frac{I^{x}M^{x}}{rvE(x)}\Big]Q . \tag{2.18}$$

Under a quasi-factorial structure with n factors $A_1,A_2,\ldots,A_n$ such that the number of levels of factor A_i is m_i, Kurkjian and Zelen [1963] denoted the quantity $E(x_1,x_2,\ldots,x_n)$ as the efficiency factor associated with the estimate of the generalized interaction $A_1^{x_1},A_2^{x_2},\ldots,A_n^{x_n}$, where $x_i = 0$ or 1 for $i = 1,2,\ldots,n$, and $A_i^0 = 1$, $A_i^1 = A_i$.

Since $\mathrm{Cov}(Q) = C\sigma^2$, we obtain (from 2.18):

$$\mathrm{Cov}(\hat{t}) = \left[{\sum}' \frac{I^x M^x}{rvE(x)} \right] \sigma^2 .$$

2.3. PAB type $k \times b$ rectanglar designs.

Consider a PA type incomplete block design with $v = \prod_{i=1}^{n} m_i$ treatments in b blocks such that each block contains k of the treatments and every treatment is replicated r times. For the purpose of obtaining a $k \times b$ rectangular design, it is desirable to regard the design as an array with k rows and b columns, where the entries in the array consist of the treatment numbers.

Define the matrices $N = (n_{ij})$ and $\tilde{N} = (n_{ih})$ to be of dimensions $v \times b$ and $v \times k$ respectively where n_{ij} = number of times treatment i occurs in block (or column); $n_{ij} = 0$ or 1, and $\tilde{n}_{ih}$ = number of times treatment i occurs in row h. Zelen and Federer [1964] denoted N as the column incidence matrix and $\tilde{N}$ as the row incidence matrix. Let $Y_{jh}(j = 1,2,\ldots,b;\ h = 1,2,\ldots,k)$ denote the measurement made in the j^{th} block and h^{th} row. When treatment i occurs in column or block j and row h, the random variable Y_{jh} will be assumed to have the expected value,

$$E\left\{Y_{jh}\right\} = \mu + t_i + b_j + \gamma_h . \tag{2.19}$$

where μ is a constant, and t_i, b_j, and γ_h are the fixed effects associated respectively with the treatments, blocks (columns), and rows. These parameters are taken to satisfy the restraints $\sum_{i=1}^{v} t_i = 0$, $\sum_{j=1}^{b} b_j = 0$, and $\sum_{h=1}^{k} \gamma_h = 0$.

Furthermore, we shall assume that $\left\{Y_{jh}\right\}$ are uncorrelated and have common variance σ^2 . Then, it is known that the reduced normal equations for estimating the treatment effect vector $t = (t_1, t_2, \ldots, t_v)'$ may be written as

$$\tilde{C}\hat{t} = \tilde{Q} , \tag{2.20}$$

where

$$\begin{aligned}
&\tilde{C} = rI_v - NN'/k - \tilde{N}\tilde{N}'/b + J_v\left(\frac{r}{v}\right) . \\
&\tilde{Q} = T - NB/k - \tilde{N}R/b + 1(g/v); \\
&T = (T_1, T_2, \ldots, T_v)', \ T_i = \text{total for treatment } i; \qquad (2.21) \\
&B = (B_1, B_2, \ldots, B_b)', \ B = \sum_{h=1}^{k} Y_{jh} = \text{total for } j^{th} \text{ block}; \\
&R = (R_1, R_2, \ldots, R_k)', \ R_k = \sum_{j=1}^{b} Y_{jh} = \text{total for } h^{th} \text{ row}; \\
&1 = \text{a } v \times 1 \text{ column vector consisting of ones; and} \\
&g = \text{total for all observations.}
\end{aligned}$$

The solution of (2.20) is $\hat{t} = \tilde{C}^{+}Q$, where $\tilde{C}^{+}$ is a generalized inverse matrix of $\tilde{C}$.

Since we are concerned with the PA type incomplete block design, this design has Property (A) in columns, i.e.,

(A)
$$NN' = \sum_{\delta} h(\delta)D^{\delta}. \tag{2.22}$$

Zelen and Federer [1964] state that a design will be said to have Property (B) which is associated with the row incidence matrix $\tilde{N}$, if

(B)
$$\tilde{N}\tilde{N}' = \sum_{\delta} \tilde{h}(\delta)D^{\delta} , \tag{2.23}$$

where $\tilde{h}(\delta) = \tilde{h}(\delta_1, \delta_2, \ldots, \delta_n)$ denotes known constants. When Property (A) and Property (B) both hold, we have

$$\tilde{C} = rI_v - NN'/k - \tilde{N}\tilde{N}'/b + J\left(\frac{r}{v}\right) = \sum_{\delta} \tilde{g}(\delta)D^{\delta}, \tag{2.24}$$

where

$$\tilde{g}(\delta) = \begin{cases} r - h(\delta)/k - h(\delta)/b & \text{for } \delta = (0,0,\ldots,0) \\ -\ [h(\delta)/k + \tilde{h}(\delta)/b] & \text{for } \delta \neq (0,0,\ldots,0),\ (1,1,\ldots,1) \\ -\ [h(\delta)/k + \tilde{h}(\delta)/b - \frac{r}{v}] & \text{for } \delta = (1,1,\ldots,1)\ . \end{cases}$$

Consequently, the reduced normal equation (2.20) takes the form

$$\Big(\sum_{\delta} \tilde{g}(\delta) D^{\delta} \Big) \hat{t} = \tilde{Q} \qquad (2.25)$$

<u>A</u> k × b <u>rectangular</u> <u>design</u> <u>will</u> <u>be</u> <u>said</u> <u>to</u> <u>have</u> <u>Properties</u> (A) <u>and</u> (B) <u>or</u> <u>will</u> <u>be</u> <u>called</u> <u>a</u> PAB <u>type</u> k × b <u>rectangular</u> <u>design</u> <u>if</u> <u>the</u> <u>design</u> <u>has</u> <u>both</u> <u>Property</u> (A) <u>and</u> <u>Property</u> (B). Suppose that the design is a PAB type rectangular design and is connected both by rows and columns separately, then from theorem 2.1, we may obtain the following solution (This solution has been given by Zelen and Federer [1964] using a quasi-factorial structure of the treatments.), i.e.,

$$\hat{t} = \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\cdots+x_n=s} \frac{\prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i}}{rv\tilde{E}(x_1,x_2,\ldots,x_n)} \right\} \tilde{Q} = \Big[\sum{}' \frac{I^x M^x}{rv\tilde{E}(x)} \Big] \tilde{Q} \qquad (2.26)$$

where $r\tilde{E}(x_1,x_2,\ldots,x_n) = \sum_{s=0}^{n-1} \Big(\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} \tilde{g}(\delta_1,\delta_2,\ldots,\delta_n) \prod_{i=1}^{n} m_i^{(1-x_i)\delta_i} (1-x_i\delta_i) \Big)$.

Note that $\tilde{C}J_v = 0$.

Since $\mathrm{Cov}(\tilde{Q}) = \tilde{C}\sigma^2$, we obtain the covariance matrix from (2.26) as:

$$\mathrm{Cov}(\hat{t}) = \Big[\sum{}' \frac{I^x M^x}{rv\tilde{E}(x)} \Big] \sigma^2 \ .$$

3. Efficiency of PA Type or PAB Type Designs.

3.1. Introduction

Suppose that we are going to test the following hypothesis in the PA type experiment, i.e.,

$$H_o: t_1 = t_2 = \ldots = t_v = 0. \tag{3.1}$$

Then, the statistic used for testing hypothesis (3.1) is given by

$$F = \hat{t}'C\hat{t}/\hat{\sigma}^2(v-1)$$

where C is a v × v matrix defined in section 2 and $\hat{\sigma}^2$ is an estimated error variance with n_e degrees of freedom. (In the case of PAB type experiment, $F = \hat{t}'\tilde{C}\hat{t}/\hat{\sigma}^2(v-1)$. The statistic has the F-distribution with v-1 and n_e degrees of freedom, and the critical region for testing the hypothesis (3.1) is given by the inequality

$$F \geq F_o , \tag{3.2}$$

where the constant F_o is determined so that the probability that $F \geq F_o$ is equal to the level of significance we wish to have.

It is known that the power function of the critical region (3.2) depends only on the single parameter

$$\phi = \frac{1}{\sigma^2}t'Ct . \tag{3.3}$$

Furthermore, this power function is monotonically increasing in ϕ . The greater the value of ϕ the more desirable is the choice of the matrix C; this matrix depends on experimental arrangement. However, a difficulty arises in that no experimental arrangement can be found for which ϕ becomes a maximum irrespective of the values of the unknown parameters $t_1, t_2, \ldots, t_v$ if $v > 1$. Hence, if $v > 1$, we have to be satisfied with some compromise solution. For this purpose, Wald [1943] considered the unit sphere

$$t_1^2 + t_2^2 + \ldots + t_v^2 = 1 ,$$

in the space of the parameters $t_1, t_2, \ldots, t_v$. It is known that if the smallest nonzero characteristic root of C is ρ_{min}, then ρ_{min} = minimum value of $\sigma^2\phi$ on the unit sphere. In a PA type design, if the design is a connected design, then the rank of C is v-1, and the matrix C may have v-1 nonzero characteristic roots. Under the unit sphere $\Sigma t_i^2 = 1$, the solution of maximizing the smallest characteristic root of C appears to be a very reasonable one to obtain maximum power. However, for the sake of certain mathematical simplifications, Wald [1943] proposed maximizing the product of the (v-1) nonzero characteristic roots of C, and he defined a measure of the design as follows: Denote by C_o the minimum value of the product of the nonzero characteristic roots of C in all possible designs for given v, k, and r. Then, the ratio $C_o / \prod_{i=1}^{v-1} \rho_i$ is called the efficiency of the design of the statistical investigation for testing hypothesis (3.1). If its efficiency is equal to one, the design is said to be optimum.

Many commonly employed symmetrical designs such as BIB (balanced incomplete block) designs, latin squares, Youden squares, etc., were shown to have optimum properties among the classes of designs (Kiefer [1958]). This was represented as an extension of a property first proved by Wald [1943]; a similar property was demonstrated by Ehrenfeld [1955].

All symmetrical designs such as BIB designs, latin squares, Youden squares, etc., are PA type or PAB type designs (see Zelen and Federer [1964]). If one is given v, k, and r and if there exists a symmetrical design, this is the optimum design in the class of designs. In the case of many other PA type or

PAB type designs, however, it may be difficult to state general properties of an optimum PA type or PAB type design in the class of designs for specified v, k, and r. In these cases, the following concept of a measure of efficiency is considered useful: Denote by C^* the value of the geometric mean of the nonzero characteristic roots of C in the BIB design for some r^*, the number of replicates of each treatment for a given v and k; we define a measure of efficiency of the PA type incomplete block design given v, k and r as follows:

$$\frac{\left(\prod_{i=1}^{v-1} \rho_i\right)^{\frac{1}{v-1}} r^*}{C^* r} \tag{3.4}$$

where $\rho_1, \rho_2, \ldots, \rho_{v-1}$ are the characteristic roots of C in the PA type design.

In the case of the PAB type rectangular designs for a particular v, k, and r, let $\tilde{C}^*$ be the value of the geometric mean of the nonzero characteristic roots of a row-generalized Youden square for some $r = r^*$ given v and k. We define a measure of efficiency of the PAB type rectangular designs given v, k, and r as follows:

$$\frac{\left(\prod_{i=1}^{v-1} \rho_i\right)^{\frac{1}{v-1}} r^*}{\tilde{C}^* r}, \tag{3.5}$$

where $\rho_1, \rho_2, \ldots, \rho_{v-1}$ are the characteristic roots of $\tilde{C}$ in the PAB type design.

[Note] A row-generalized Youden square means that the column incidence matrix type is BIB type and row incidence matrix type is randomized complete block type, but each row may contain the set of all treatments more than once.

3.2. Characteristic roots of C and $\tilde{C}$.

Define $A_i' = (A_i^{(0)'}, A_i^{(1)'}, \ldots, A_i^{(m_i-1)'})$ to be an $m_i \times m_i$ matrix such that $A_i A_i' = I_{m_i}$ and $A_i^{(0)} = \frac{1}{\sqrt{m_i}} 1_{m_i}'$.

Then,

$$A_i^{(z_i)} D_i^{\delta_i} = \begin{cases} A_i^0 & \text{for } z_i = 0,\ \delta_i = 0 \\ m_i A_i^{(1)} & \text{for } z_i = 0,\ \delta_i = 1 \\ A_i^{(0)} & \text{for } z_i = 1,\ \delta_i = 0 \\ 0 & \text{for } z_i = 1,\ \delta_i = 1 \\ A_i^{(2)} & \text{for } z_i = 2,\ \delta_i = 0 \\ \vdots & \quad\vdots \\ A_i^{(m_i-1)} & \text{for } z_i = m_i-1,\ \delta_i = 0 \\ 0 & \text{for } z_i = m_i-1,\ \delta_i = 1 \end{cases} .$$

Then,

$$A_i^{(z_i)} D_i^{\delta_i} = m_i^{(1-x_i)\delta_i} (1-x_i\delta_i) A_i^{(z_i)}, \tag{3.6}$$

where $x_i = 0$ if $z_i = 0$ and $x_i = 1$ otherwise.

Let $A^z = \prod_{i=1}^{n} \otimes A_i^{(z_i)}$, where $z = (z_1, z_2, \ldots, z_n)$ and $z_i = 0, 1, \ldots, m_i-1$ for all $i = 1, 2, \ldots, n$, and let $A = \prod_{i=1}^{n} \otimes A_i$, then

$$A = \prod_{i=1}^{n} \otimes \begin{bmatrix} A_i^{(0)} \\ A_i^{(1)} \\ \vdots \\ A_i^{(m_i-2)} \\ A_i^{(m_i-1)} \end{bmatrix} = \begin{bmatrix} \prod_{i=1}^{n} \otimes A_i^{(0)} \\ \prod_{i=1}^{n} \otimes A_i^{(1)} \\ \cdots \\ \prod_{i=1}^{n} \otimes A_i^{(m_i-1)} \otimes A_n^{(m_n-1)} \\ \prod_{i=1}^{n} \otimes A_i^{(m_i-1)} \end{bmatrix} = \begin{bmatrix} A^{(0,0,\ldots,0)} \\ A^{(0,0,\ldots,1)} \\ \cdots \\ A^{(m_1-1, m_1-1, \ldots m_n-2)} \\ A^{(m_1-1, m_2-1, \ldots, m_n-1)} \end{bmatrix} \tag{3.7}$$

$$\text{Let } r\rho(z_1,z_2,\dots,z_n) = \sum_{s=0}^{n-1}\Big\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\ \delta_2,\dots,\delta_n) \prod_{i=1}^{n} m_i^{(1-x_i)\delta_i}(1-x_i\delta_i)\Big\}$$

$$= rE(x_1,x_2,\dots,x_n)\ , \tag{3.8}$$

where $x_i = 0$ if $z_i = 0$, and $x_i = 1$ otherwise. Then,

$$A^{z}C = \sum_{s=0}^{n-1}\Big\{ \sum_{\delta_1+\delta_2+\dots+\delta_n=s} g(\delta_1,\delta_2,\dots,\delta_n) \prod_{i=1}^{n} \otimes\, m_i^{(1-x_i)\delta_i}(1-x_i\delta_i)A_i^{(z_i)}\Big\}$$

$$= r\rho(z_1,z_2,\dots,z_n)A^{(z_1,z_2,\dots,z_n)}\ .$$

Also,

$$AC = \begin{bmatrix} A^{(0,0,\dots,0)}C \\ A^{(0,0,\dots,1)}C \\ A^{(0,0,\dots,2)}C \\ \cdots \\ A^{(m_1-1,\ m_2-1,\dots,m_n-2)}C \\ A^{(m_1-1,\ m_2-1,\dots,m_n-1)}C \end{bmatrix} = \begin{bmatrix} 0 \\ r\rho(0,0,\dots,1)A^{(0,0,\dots,1)} \\ r\rho(0,0,\dots,2)A^{(0,0,\dots,2)} \\ \cdots \\ r\rho(m_1-1,m_2-1,\dots,m_n-2)A^{(m_1-1,m_2-1,\dots m_n-2)} \\ r\rho(m_1-1,m_2-1,\dots,m_n-1)A^{(m_1-1,m_2-1,\dots m_n-1)} \end{bmatrix}$$

Hence,

$$ACA' = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r\rho(0,0,\dots,1) & 0 & \cdots & 0 & 0 \\ 0 & 0 & r\rho(0,0,\dots,2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r\rho(m_1-1,m_2-1,\dots,m_n-2) & 0 \\ 0 & 0 & 0 & \cdots & 0 & r\rho(m_1-1,m_2-1,\dots,m_n-1) \end{bmatrix} .$$

$$\tag{3.9}$$

Note that

$$r\rho(0,0,\ldots,0,z_n) = rE(0,0,\ldots,0,1) \quad \text{for } z_n = 1,2,\ldots,m_n-1,$$

$$r\rho(0,0,\ldots,z_{n-1},0) = rE(0,0,\ldots,1,0) \quad \text{for } z_{n-1} = 1,2,\ldots,m_{n-1}-1,$$

$$\ldots \qquad\qquad\qquad \ldots$$

$$r\rho(z_1,z_2,\ldots,z_n) = rE(1,1,\ldots,1) \quad \text{for } z_i = 1,2,\ldots,m_i-1, i = 1,2,\ldots,$$

Now, we can state the following theorem:

Theorem 3.1. _Define matrix A as in (3.7); then matrix A is a characteristic vector of C and if a PA type incomplete block design is connected, the nonzero characteristic roots of C are as follows:_

$$\begin{aligned}
&m_n-1 \text{ of } rE(0,0,\ldots,0,1)\\
&m_{n-1}-1 \text{ of } rE(0,0,\ldots,1,0)\\
&\qquad\ldots\\
&m_1-1 \text{ of } rE(1,0,\ldots,0,0)\\
&(m_n-1)(m_{n-1}-1) \text{ of } rE(0,0,\ldots,1,1)\\
&\prod_{i=1}^{n}(m_i-1) \text{ of } rE(1,1,\ldots,1,1)\ .
\end{aligned} \tag{3.10}$$

The same result holds for the matrix $\tilde{C}$ with corresponding changes of notation.

3.3. Efficiency criterion for PA type and PAB type designs.

Given v and k, it may not be difficult to construct a BIB design or a row-generalized Youden square design for some $r = r^*$. In this case,

$$NN' = (r^* - \lambda)I_v + \lambda J_v, \tag{3.11}$$

where $\lambda = r^*(k-1)/(v-1)$, and

$$\tilde{N}\tilde{N}' = \frac{r^{*2}}{k}J_v\ . \tag{3.12}$$

Then,

$$C = r^{*}I_v - \frac{1}{k}NN'$$

$$= \frac{r^{*}v(k-1)}{k(v-1)} I_v - \frac{r^{*}(k-1)}{k(v-1)}J_v \text{ , since } vr^{*} = kb \qquad (3.13)$$

$$\tilde{C} = r^{*}I_v - \frac{1}{k}NN' - \frac{1}{b}\tilde{N}\tilde{N}' + \frac{r^{*}}{v}J_v$$

$$= \frac{r^{*}v(k-1)}{k(v-1)}I_v - \frac{r^{*}(k-1)}{k(v-1)}J_v \text{ ,}$$

and

$$r^{*}E(x_1,x_2,\ldots,x_n) = \frac{r^{*}v(k-1)}{k(v-1)} = r^{*}\tilde{E}(x_1,x_2,\ldots,x_n) \quad .$$

We now give the following definition for a measure of efficiency of the PA type incomplete block design or PAB type rectangular design given v, k, and r:

Efficiency of the PA type or PAB type design

$$= \frac{k(v-1)\left[\prod_{s=1}^{n} \prod_{x_1+x_2+\ldots+x_n=s} (Ex_1,x_2,\ldots,x_n)^{p(x_1,x_2,\ldots,x_n)}\right]^{\frac{1}{v-1}}}{v(k-1)} \qquad (3.14)$$

where $p(x_1,x_2,\ldots,x_n) = (m_1-1)^{x_1}(m_2-1)^{x_2}\ldots(m_n-1)^{x_n}$.

<u>Example</u>

Consider the following design for v = 6, k = 3, r = 5

1	1	2	4	5	3	5	6	2	4
4	2	6	3	4	6	1	1	3	5
6	5	3	1	2	5	3	2	4	6

$$N = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$NN' = \begin{bmatrix} 5 & 2 & 2 & 2 & 2 & 2 \\ 2 & 5 & 2 & 2 & 2 & 2 \\ 2 & 2 & 5 & 2 & 2 & 2 \\ 2 & 2 & 2 & 5 & 2 & 2 \\ 2 & 2 & 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 2 & 2 & 5 \end{bmatrix} = 3I_6 + 2J_6$$

This design is a PA type (actually a BIB) design. Since $C = 5I_6 - \frac{1}{3}(3I_6 + 2J_6) = \frac{1}{3}(12I_6 - 2J_6)$, $g(0,0) = 4$, $g(0,1) = 0$, $g(1,0) = 0$, and then $E(0,1) = \frac{4}{5}$, $E(1,0) = \frac{4}{5}$, $E(1,1) = \frac{4}{5}$. Hence, from (3.14), the efficiency of this design is

$$\frac{3(6-1)(4/5)}{6(3-1)} = 1 \quad .$$

Therefore, the above PA type incomplete block design is an optimum design given $v = 6$, $k = 3$, and $r = 5$.

Next

$$\tilde{N} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad \tilde{N}\tilde{N}' = \begin{bmatrix} 9 & 8 & 8 & 9 & 8 & 8 \\ 8 & 9 & 8 & 8 & 9 & 8 \\ 8 & 8 & 9 & 8 & 8 & 9 \\ 9 & 8 & 8 & 9 & 8 & 8 \\ 8 & 9 & 8 & 8 & 9 & 8 \\ 8 & 8 & 9 & 8 & 8 & 9 \end{bmatrix} = J_2 \otimes I_3 + 8J_6 \, .$$

Thus, the above design is a PAB type design.

Since $\tilde{C} = 5I_6 - \frac{1}{3}(3I_6 + 2J_6) - \frac{1}{10}(J_2 \otimes I_3 + 8J_6) + \frac{5}{6}J_6 = \frac{1}{30}(120I_6 - 3J_2 \otimes I_3 - 19J_6)$, $\tilde{g}(0,0) = 4$, $\tilde{g}(0,1) = 0$, $\tilde{g}(1,0) = -\frac{1}{10}$, and then $\tilde{E}(0,1) = 0.76$, $\tilde{E}(1,0) = 0.8$, $\tilde{E}(1,1) = 0.8$. Hence, efficiency of the above PAB type design

$$= \frac{3(6-1)\ [(0.76)^2(0.8)(0.8)^2]^{\frac{1}{5}}}{6(3-1)} = 0.98 .$$

In this case, the meaning of the efficiency is that the efficiency of the above design is 0.98 compared with the row-generalized Youden square for some $r = r^*$ which is optimum given $v = 6$ and $k = 3$.

4. Some Relationships Between PA Type and PBIB Designs.

4.1. Definition of basic matrix and its properties.

Suppose $v = m_1, m_2, \ldots, m_n$, where m_i are all prime numbers and $n \geqq 2$. Define a basic matrix

$$M_{1,2,\ldots,n}(\delta_0, \delta_1, \ldots, \delta_n) = \delta_0(I_v + J_v) + (-1)^{\delta_0} \prod_{i=1}^{n} \otimes D_i^{\delta_i}, \tag{4.1}$$

for $\delta_i = 0$ or 1 for all $i = 0,1,\ldots,n$, and $D_i^{\delta_i}$ has been defined in section 2. Consider the following set $S_M(1,2,\ldots,n)$:

$$S_M(1,2,\ldots,n) = \left\{M_{1,2,\ldots,n}(\delta_0, \delta_1, \ldots, \delta_n) : \delta_i = 0 \text{ or } 1 \text{ for all } i = 0,1,\ldots,n, \text{ and } \delta_0 + \delta_1 + \ldots + \delta_n \neq 0, \neq n+1, \text{ and } \delta_1 + \delta_2 + \ldots + \delta_n \neq 0\right\}. \tag{4.2}$$

It may be easily verified that the number of elements in the set $S_M(1,2,\ldots,n)$ is $2^{n+1} - 3$.

Let $m_{(1)}, m_{(2)}, \ldots, m_{(n)}$ be a permutation of $m_1, m_2, \ldots, m_n$. Then, given the permutation, we may obtain a new set $S_M((1),(2),\ldots,(n))$. The two sets $S_M(1,2,\ldots,n)$ and $S_M((1),(2),\ldots,(n))$ are not the same set if all m_i, $i = 1,2,\ldots,n$, are not equal to a constant number. Thus, we may obtain $n!$ different sets of S_M

Proof: Let $M_{1,2,\ldots,n-1}(\delta_0,\delta_1,\ldots,\delta_{n-1}) = (\lambda_{ij})$, $i,j = 1,2,\ldots,v_{n-1}$, where $v_{n-1} = \prod_{i=1}^{n-1} m_i$, and let $D_n^{\delta_n} = (d_{ij})$, $i,j = 1,2,\ldots,m_n$, then the $((k-1)m_n + h)^{th}$ row of $M_{1,2,\ldots,(n-1)}(\delta_0,\delta_1,\ldots,\delta_{n-1}) \otimes D_n^{\delta_n}$ is

$$\lambda_{k1}(d_{h1},d_{h2},\ldots,d_{hm_n}),\ldots,\lambda_{kh}(d_{h1},d_{h2},\ldots,d_{hm_n}),\ldots,\lambda_{kv_{n-1}}(d_{h1},d_{h2},\ldots,d_{hm_n}),$$

where $1 \le k \le v_{n-1}$, $1 \le h \le m_n$.

Therefore,

$$\nu\{M_{1,2,\ldots,n-1}(0,\delta_0,\ldots,\delta_{n-1}) \otimes D_n^{\delta_n}\} = \Big(\sum_{j=1}^{m_n} d_{hj}\Big)\Big(\sum_{\substack{j'=1\\ j'\neq k}}^{v_{n-1}-1} \lambda_{kj'}\Big) + \lambda_{kk}\Big(\sum_{\substack{j=1\\ j\neq k}}^{m_n} d_{kj}\Big)$$

$$= m_n^{\delta_n}\nu(0,\delta_1,\ldots,\delta_{n-1}) + m_n^{\delta_n} - 1, \text{ since } \lambda_{kk}d_{hh} = 1 .$$

Next,

$$\begin{aligned}
\nu(1,\delta_1,\ldots,\delta_n) &= \nu\{M_{1,2,\ldots,n}(1,\delta_1,\ldots,\delta_n)\} \\
&= \nu\{I_v + J_v\} - \nu\{M_{1,2,\ldots,n}(0,\delta_1,\ldots,\delta_n)\} \\
&= \nu(0,1,\ldots,1) - \nu(0,\delta_1,\ldots,\delta_n) \\
&= v - 1 - \nu(0,\delta_1,\ldots,\delta_n) .
\end{aligned}$$

Lemma 4.2.

$$\nu(\delta_0,\delta_1,\ldots,\delta_n) = \delta_0\Big(\prod_{i=1}^{n} m_i - 1\Big) + (-1)^{\delta_0}\Big(\prod_{i=1}^{n} m_i^{\delta_i} - 1\Big) \tag{4.5}$$

Proof:

To prove formula (4.5) we have for $n = 1$:

$M_1(0,1) = J_{m_1}$, $\nu(0,1) = m_1 - 1$, and

$M_1(1,0) = I_{m_1} + J_{m_1} - I_{m_1} = J_{m_1}$; also, $\nu(1,0) = m_1 - 1$;

and formula (4.5) is clearly true for $n = 1$.

Now assume that for some k,

$$\nu(0,\delta_1,\ldots,\delta_k) = \prod_{i=1}^{k} m_i^{\delta_i} - 1 ,$$

and

$$\nu(1,\delta_1,\ldots,\delta_k) = \prod_{i=1}^{k} m_i - 1 - (\prod_{i=1}^{k} m_i^{\delta_i} - 1) .$$

Then, from (4.3)

$$\begin{aligned}\nu(0,\delta_1,\ldots,\delta_{k+1}) &= m_{k+1}^{\delta_{k+1}}\nu(0,\delta_1,\ldots,\delta_k) - (m_{k+1}^{\delta_{k+1}} - 1)\\ &= \prod_{i=1}^{k+1} m_i^{\delta_i} - m_{k+1}^{\delta_{k+1}} + m_{k+1}^{\delta_{k+1}} - 1\\ &= \prod_{i=1}^{k+1} m_i^{\delta_i} - 1 ,\end{aligned}$$

and from (4.4)

$$\begin{aligned}\nu(1,\delta_1,\ldots,\delta_k,\delta_{k+1}) &= \nu(0,1,\ldots,1) - \nu(0,\delta_1,\ldots,\delta_k,\delta_{k+1})\\ &= \left(\prod_{i=1}^{k+1} m_i - 1\right) + (-1)^1\left(\prod_{i=1}^{k+1} m_i^{\delta_i} - 1\right) .\end{aligned}$$

Thus, the formula is true for k + 1. By the principle of induction, this proves formula (4.5) for all natural numbers n.

4.2. Some relationships between PA type and PBIB designs.

Let N be an incidence matrix of a PA type incomplete block design with $v = \prod_{i=1}^{n} m_i$ treatments in blocks such that each block contains k experimental units and every treatment is replicated r times.

Suppose that the matrix NN' of the design has the following form:

$$NN' = \alpha I_v + M_{1,2,\ldots,n}(\delta_0,\delta_1,\ldots,\delta_n) \qquad (4.6)$$

where α is a non-negative integer. Then the following properties of the design hold:

(a). Since the off-diagonal elements λ_{ij}, $i \neq j$, of $M(\delta_o, \delta_1, \ldots \delta_n)$ are 0 or 1, any two treatments are either first associates ($\lambda_1 = 0$) or second associates ($\lambda_2 = 1$).

(b). Since $\sum\limits_{\substack{i \\ i \neq j}} \lambda_{ij} = \sum\limits_{\substack{j \\ j \neq i}} \lambda_{ij} = \nu(\delta_0, \delta_1, \ldots, \delta_n)$ = constant for a given $(\delta_0, \delta_1, \ldots, \delta_n)$, each treatment has exactly $\nu(\delta_0, \delta_1, \ldots, \delta_n)$ second associates and $(v-1-\nu(\delta_0, \delta_1, \ldots \delta_n))$ first associates.

Next, given any two treatments which are i^{th} associates, let $p^i_{jk}(i,j,k = 1,2)$ be the number of treatments common to the j^{th} associate of the first and the k^{th} associate of the second. For a design which has the property given by (4.6) and for $\lambda_{ii}\lambda_{ji} = 0$, from the matrix $M(\delta_0, \delta_1, \ldots, \delta_n) = (\lambda_{ij})$, the i^{th} treatment and the j^{th} treatment are first associates; also, if $\lambda_{ii}\lambda_{ji} = 1$, then these two treatments are second associates.

Now, suppose that $M(\delta_0, \delta_1, \ldots, \delta_n) = M(0, \delta_1, \delta_2, \ldots, \delta_n)$; then, since $M(\delta_0, \delta_1, \ldots, \delta_n)$ is symmetric,

$$\begin{aligned} M(\delta_0, \delta_1, \ldots, \delta_n)M(\delta_0, \delta_1, \ldots, \delta_n)' &= (M(\delta_0, \delta_1, \ldots, \delta_n))^2 \\ &= \Big(\prod_{i=1}^{n} \otimes D_i^{\delta_i}\Big)^2 \\ &= \prod_{i=1}^{n} \otimes \Big(D_i^{\delta_i}\Big)^2 \\ &= \prod_{i=1}^{n} m_i^{\delta_i} \prod_{i=1}^{n} \otimes D_i^{\delta_i} = \prod_{i=1}^{n} m_i^{\delta_i}(\lambda_{ij}), \end{aligned} \tag{4.7}$$

i.e.,

$$\sum_{\substack{k=1 \\ i<j}}^{v} \lambda_{ik}\lambda_{jk} = \begin{cases} 0 & \text{if } \lambda_{ij} = 0 \text{ or } \lambda_{ii}\lambda_{ij} = 0 \\ \prod\limits_{i=1}^{n} m_i^{\delta_i} & \text{if } \lambda_{ij} = 1 \text{ or } \lambda_{ii}\lambda_{ij} = 1 \end{cases} . \tag{4.8}$$

Consider the following pairs of numbers:

$$(\lambda_{i1},\lambda_{j1}),\ (\lambda_{i2},\lambda_{j2}),\ldots,(\lambda_{ik},\lambda_{jk}),\ldots,(\lambda_{iv},\lambda_{jv}), \qquad (4.9)$$

then, $\sum_{\substack{k=1\\i<j}}^{v} \lambda_{ik}\lambda_{jk} = 0$ means that there is no pair of numbers such that

$(\lambda_{ik},\lambda_{jk}) = (1,1)$. Therefore, we obtain $p'_{22} = 0$, and this is independent of the pair of treatments with which we start.

$\sum_{\substack{k=1\\i<j}}^{v} \lambda_{ik}\lambda_{jk} = 0$ also means that there are $\nu(0,\delta_1,\ldots,\delta_n)$ pairs of numbers such that

$(\lambda_{ik},\lambda_{jk}) = (0,1)$ for $k \neq i$, and $k \neq j$, and this means

$$p'_{12} = \nu(0,\delta_1,\ldots,\delta_n) = p'_{21}\ .$$

Thus, we have

$$p'_{11} = v - 2 - 2\nu(0,\delta_1,\ldots,\delta_n) = v - 2 - 2p'_{12}\ .$$

Similarly, if $\lambda_{ij} = 1$, then $\sum_{\substack{k=1\\i<j}}^{v} \lambda_{ik}\lambda_{jk} = \prod_{i=1}^{n} m_i^{\delta_i} - 2$, and this means

$p^2_{22} = \prod_{i=1}^{n} m_i^{\delta_i} - 2$, $p^2_{12} = \nu(0,\delta_1,\ldots,\delta_n) - \left(\prod_{i=1}^{n} m_i^{\delta_i} - 1\right) = p^2_{21}$, and

$p^2_{11} = v - 2 - p^2_{12} - p^2_{22}$.

Next, suppose that $M(\delta_0,\delta_1,\ldots,\delta_n) = M(1,\delta_1,\ldots,\delta_n)$. Then

$$(M_{1,2,\ldots,n}(1,\delta_1,\ldots,\delta_n))^2 = (I_v + J_v - \prod_{i=1}^{v} \otimes D_i^{\delta_i})^2$$

$$= I_v + (v-2)\nu(0,\delta_1,\ldots,\delta_n)J_v + \left(\prod_{i=1}^{n} m_i^{\delta_i} - 2\right)\left(\prod_{i=1}^{n} \otimes D_i^{\delta_i}\right), \qquad (4.9)$$

since $J_v\left(\prod_{i=1}^{n} \otimes D_i^{\delta_i}\right) = (\nu(0,\delta_1,\ldots,\delta_n) + 1)\, J_v$.

Let $M_{1,2,\ldots,n}(1,\delta_1,\ldots,\delta_n) = (\lambda^*_{ij})$, then it may be easily verified that

$$\lambda^*_{ij}\lambda^*_{ji} = \begin{cases} 0 & \text{if } \lambda_{ij} = 1 \\ 1 & \text{if } \lambda_{ij} = 0 \end{cases}$$

where λ_{ij} is the $(i,j)^{th}$ element of $M(0,\delta_1,\ldots,\delta_n)$, and

$$\sum_{\substack{k=1 \\ i<j}}^{v} \lambda^*_{ik}\lambda^*_{jk} = \begin{cases} v - 2\nu(0,\delta_1,\ldots,\delta_n) + \prod_{i=1}^{n} m_i^{\delta_i} - 2 & \text{if } \lambda^*_{ij} = 0 \\ v - 2\nu(0,\delta_1,\ldots,\delta_n) & \text{if } \lambda^*_{ij} = 1 \end{cases} \quad (4.10)$$

In this case, if $\lambda^*_{ij} = 0$ then the i^{th} treatment and j^{th} treatment are first associates and

$$\sum_{\substack{k \\ i<j}}^{v} \lambda^*_{ik}\lambda^*_{jk} = \text{number of the pair of numbers } (\lambda^*_{ik}, \lambda^*_{jk}) \text{ such that}$$

$(\lambda^*_{ik}, \lambda^*_{jk}) = (1,1)$. From (4.10)

$$p^1_{22} = v - 2\nu(0,\delta_1,\ldots,\delta_n) + \prod_{i=1}^{n} m_i^{\delta_i} - 2 ,$$

Then, $p^1_{12} = \nu(1,\delta_1,\ldots,\delta_n) - p^1_{22} = v - 1 - \nu(0,\delta_1,\ldots,\delta_n) - p^1_{22} = p^1_{21}$, and

$p^1_{11} = v - 2 - 2p^1_{12} - p^1_{22}$. Similarly, if $\lambda^*_{ij} = 1$,

$$p^2_{22} = \sum_{k=1}^{v} \lambda^*_{ik}\lambda^*_{jk} - 2 = v - 2\nu(0,\delta_1,\ldots,\delta_n) - 2, \quad p^2_{12} = \nu(1,\delta_1,\ldots,\delta_n) - 1 - p^2_{22}$$

$$= v - 2 - \nu(0,\delta_1,\ldots,\delta_n) - p^2_{22} = p^2_{21}, \text{ and } p^2_{11} = v - 2 - 2p^2_{12} - p^2_{22} .$$

Now, we can state the following property of the above PA type designs, i.e.,

(c) Given any two treatments which are i^{th} associates, the number of treatments common to the j^{th} associates of the first and k^{th} associates of the second is p^i_{jk} and is independent of the pair of the treatments with which we start. Also, $p^i_{jk} = p^i_{kj}$ $(i,j,k = 1,2)$.

It is easy to extend the above results to the following form of NN' :

$$NN' = c_0 I_v + c_1 J_v + c_2 M_{1,2,\ldots,n}(\delta_0, \delta_1, \ldots, \delta_n)$$

where c_0, c_1, and c_2 are non-negative integers. In this case, $\lambda_1 = c_1$, and $\lambda_2 = c_1 + c_2$. However, the values of p^i_{jk} will not be changed from the case of $NN' = \alpha I_v + M_{1,2,\ldots,n}(\delta_0, \delta_1, \ldots, \delta_n)$.

Finally, we can state the following theorem.

<u>Theorem 4.1.</u> <u>In a PA type incomplete block design, if</u>

$$NN' = c_0 I_v + c_1 J_v + c_2 M_{1,2,\ldots,n}(\delta_0, \delta_1, \ldots, \delta_n) \tag{4.11}$$

<u>for</u> $v = \prod_{i=1}^{n} m_i$, <u>where</u> c_0, c_1, <u>and</u> c_2 <u>are non-negative integers, then the design is a partially balanced incomplete block design with two associate classes</u> $(\lambda_1 = c_1,\ \lambda_2 = c_1 + c_2)$.

The concept of this theorem may be extended to any PA type incomplete block design; a generalized definition of PBIB designs is needed.

<u>Examples</u>

(1). $v = 8,\ k = 3,\ b = 8,\ r = 3.$

1	2	3	4	5	6	7	8
3	4	5	6	7	8	1	2
4	5	6	7	8	1	2	3

In this case,

$$NN' = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} = 3I_8 - J_2 \otimes I_2 + J_8 = 2I_8 + M_{1,2,3}(1,1,00)$$

$$c_0 = 2, \quad c_1 = 0, \quad c_2 = 1$$

$$\lambda_1 = 0, \quad \lambda_2 = 1$$

$p'_{22} = v - 2\nu(0,\delta_1,\dots,\delta_n) + \prod_{i=1}^{n} m_i^{\delta_i} - 2 = 8 - 2\nu(0,1,0,0) + 2 - 2 = 8 - 2(1) + 2 - 2 =$

$p'_{12} = p'_{21} = v - 1 - \nu(0,1,0,0) - p'_{22} = 8 - 1 - 1 - 6 = 0,$

$p'_{11} = v - 2 - 2p'_{12} - p'_{22} = 8 - 2 - 2(0) - 6 = 0,$

$p^2_{22} = v - 2\nu(0,1,0,0) - 2 = 8 - 2(1) - 2 = 4,$

$p^2_{12} = p^2_{21} = v - 2 - \nu(0,1,0,0) - p^2_{22} = 1$, and $p^2_{11} = v - 2 - 2p^2_{12} - p^2_{22} = 0.$

(2) $v = 6,\ k = 4,\ b = 6,\ r = 4$

1	2	3	4	5	6
2	3	1	6	4	5
3	1	2	5	6	4
4	5	6	1	2	3

In this case,

$$NN' = \begin{bmatrix} 4 & 3 & 3 & 2 & 2 & 2 \\ 3 & 4 & 3 & 2 & 2 & 2 \\ 3 & 3 & 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 4 & 3 & 3 \\ 2 & 2 & 2 & 3 & 4 & 3 \\ 2 & 2 & 2 & 3 & 3 & 4 \end{bmatrix} = I_6 + I_2 \otimes J_3 + 2J_6 = I_6 + 2J_6 + M_{1,2}(0,0,1)$$

$$c_0 = 1, \quad c_1 = 2, \quad c_2 = 1$$

$$\lambda_1 = 2, \quad \lambda_2 = c_1 + c_2 = 3$$

$p'_{22} = 0$, $p'_{12} = p'_{21} = \nu(0,0,1) = 2$, $p'_{11} = v - 2 - 2p'_{12} = 6 - 2 - 2(2) = 0$,
$p^2_{22} = 2^0.3^1 - 2 = 1$, $p^2_{12} = p^2_{21} = \nu(0,0,1) - (2^0.3^1 - 1) = 2 - (3-1) = 0$, and
$p^2_{11} = 6 - 2 - p^2_{12} - p^2_{22} = 6 - 2 - 0 - 1 = 3$.

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